An Exploration of Continued Fractions

Joseph Wakeling

Note

This essay is divided into three sections. The first deals with the basic theory of continued fractions – the definitions and a few fundamental results. The second deals with its applications in working with rational and irrational numbers, in particular in relation to finding rational approximations for irrationals. The final section gives some interesting examples of applications of continued fractions.

In writing this essay, I've followed a general principle of, where the proofs in my source reading were sketchy or inadequate, to try and fill in those missed-out details, and where they were complete, to try and gloss over the majority of the number-crunching and give instead the *idea* of the proof. Naturally, this principle has been somewhat bent and/or broken on occasions, but this was my aim!

1. What is a continued fraction?

DEFINITION 1.1

A *finite continued fraction* is a number of the form

$$\theta = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_n}}}} \tag{1}$$

Where the a_i are integers.

For compactness' sake, we have two alternative forms of notation for this:

$$\theta = a_0 + \frac{1}{a_1 +}\frac{1}{a_2 +}\cdots\frac{1}{a_{n-1} +}\frac{1}{a_n} \tag{2}$$

or, if we're really pushed for space,

$$\theta = [a_0, a_1, \ldots, a_n] \tag{3}$$

What's so special about this? Isn't it just a complicated way of writing out a rational number? (Recall, the rationals are closed under addition and multiplication.) Well…

For the expression given in (1), yes. But what if the expression *doesn't* terminate – in other words, if it is *infinitely* continued?[1]

$$\theta = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_i + \ddots}}}} \tag{4}$$

What happens now? Is *this* a rational number? The question isn't easy to answer at first glance, because while it's simple logic that if θ is irrational its continued fraction will be infinite, that doesn't mean that all infinite continued fractions are irrational[2]. Maybe, given the things we already know about rationals, there are other things we want to know – for example, we know that a number is rational if and only if its decimal expansion is recurrent. What can we say about a number θ if its continued fraction is recurrent? Or what if it isn't?

Let's begin our exploration of these questions by looking at the problem in an alternative way: given a number θ, how can we find its continued fraction?

First, let's remind ourselves of the notation $[x]$, meaning *the largest integer less than x*. *Don't* confuse this with the shorthand notation for a continued fraction! Now, we're ready to begin.

ALGORITHM 1.2

The *Continued Fraction Algorithm* allows us to find the continued fraction for any real number θ.

Given θ, we set $a_0 = [\theta]$. Now, check: if $\theta = a_0$, we're done. If $\theta \neq a_0$, write $\theta = a_0 + \frac{1}{\theta_1}$.

Set $a_1 = [\theta_1]$. Check again: if $\theta_1 = a_1$, stop. Otherwise, we write $\theta_1 = a_1 + \frac{1}{\theta_2}$ (and hence, $\theta = a_0 + \cfrac{1}{a_1 + \frac{1}{\theta_2}}$).

---

[1] The compact notation for this is either $\theta = a_0 + \frac{1}{a_1 +}\frac{1}{a_2 +}\cdots$ or simply $\theta = [a_0, a_1, a_2, \ldots]$.

[2] Continued fraction terminates $\Rightarrow \theta$ is rational, so consequently θ is irrational $\Rightarrow$ the continued fraction is infinite. But to assume that the continued fraction is infinite $\Rightarrow \theta$ is irrational is committing the 'cow error', the all-time classic mathematical slip-up!

… and so on and so on: set $a_i = [\theta_i]$; if $\theta_i \neq a_i$, write $\theta_i = a_i + \frac{1}{\theta_{i+1}}$; and repeat. The result will be a continued fraction like those described above, and we call this *the continued fraction of* θ; the a_i we refer to as the (partial) quotients of θ.

Note that the a_i are always positive for $i \geq 1$, as by definition $[x] \leq x$, and so the 'remainder' $\frac{1}{\theta_i}$ at each step will always be positive.

The result of this algorithm is a sequence of rational numbers $\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$ tending to θ as $n \to \infty$. (We think of these rationals in lowest terms, so p_n, q_n are uniquely determined, relatively prime integers.) We call these rationals $\frac{p_n}{q_n}$ the *convergents* to θ, and we will now prove that they do indeed converge as described.

We'll begin with the following two Lemmas:

LEMMA 1.3

The convergents $\frac{p_n}{q_n}$ are generated recursively by the equations

$$p_n = a_n p_{n-1} + p_{n-2} \qquad q_n = a_n q_{n-1} + q_{n-2} \tag{5}$$

where $p_0 = a_0$, $q_0 = 1$, $p_1 = a_0 a_1 + 1$ and $q_1 = a_1$.

Proof.

A slightly tricksy induction-based proof. First, note that the equations hold for the case $n = 2$. Now, suppose that the equations hold for $n = m - 1 \geq 2$; we want to show that they must also hold for $n = m$. We'll leave out the details as it's mostly just number-crunching (symbol-crunching?); the trick is to define the finite continued fraction $\frac{p'_j}{q'_j} = [a_1, a_2, \ldots, a_{j+1}]$ and note that, first, by the induction hypothesis, p'_j and q'_j are defined by the equations above when $j = m - 1$, and second, we have $\frac{p_j}{q_j} = a_0 + \frac{p'_{j-1}}{q'_{j-1}}$, so we can define p_j and q_j in terms of p'_{j-1} and q'_{j-1}. Put $j = m$ and the result follows. ■

LEMMA 1.4

The following equality always holds:

$$p_n q_{n+1} - p_{n+1} q_n = (-1)^{n+1} \tag{6}$$

Proof.

By induction. Note that equation (6) holds for $n = 0$. Now, assume it holds when $n = m$ and show that this implies it holds for $n = m + 1$ by using the equations proved in **lemma 1.3** to express p_{m+2} and q_{m+2} in terms of p_m, p_{m+1} and q_m, q_{m+1} respectively. ■

This enables us to prove

THEOREM 1.5

The convergents $\frac{p_n}{q_n}$ converge to θ as $n \to \infty$.

Proof.

By definition we have

$$\theta = [a_0, a_1, \ldots, a_n, \theta_{n+1}] \quad (7)$$

and since $a_{n+1} = [\theta_{n+1}] \leq \theta_{n+1}$, we have $0 < \frac{1}{\theta_{n+1}} \leq \frac{1}{a_{n+1}}$. Hence, θ lies between $\frac{p_n}{q_n}$ $(= [a_0, \ldots, a_n])$ and $\frac{p_{n+1}}{q_{n+1}}$ $(= [a_0, \ldots, a_{n+1}])$.

By **lemma 1.4** we have $p_n q_{n+1} - p_{n+1} q_n = (-1)^{n+1}$ and hence we have

$$\left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| = \frac{1}{q_n q_{n+1}} \quad (8)$$

and since θ lies between $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$, it follows that

$$\left| \theta - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n q_{n+1}} \quad (9)$$

The equations proved in **lemma 1.3** show that $\{q_n\}$ is an increasing sequence[3], and so it follows that the RHS of equation (9) tends to zero as $n \to \infty$.

Hence, $\frac{p_n}{q_n} \to \theta$ as $n \to \infty$. ■

This result also gives us what we need to answer one of our original questions: what can we say about a real number θ if its continued fraction is infinite? First, we need the following Lemma:

LEMMA 1.6

For any real number θ, there are infinitely many rational numbers $\frac{p}{q}$ satisfying the inequality

$$\left| \theta - \frac{p}{q} \right| < \frac{1}{q^2} \quad (10)$$

if and only if θ is irrational.

Equivalently, there are only finitely many such $\frac{p}{q}$ if and only if θ is rational.

---

[3] This can be seen to be true as $q_0 = 1$, $q_1 = a_1 > 0$, and hence all the numbers generated by the recursive equation for q_n will be positive.

Proof.

Suppose θ is irrational. Then, by the equality (9) that we proved in the proof of **theorem 1.5**, we have $\left|\theta - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n q_{n+1}}$ for all the convergents $\frac{p_n}{q_n}$. Since (using the recursive equations proven in **lemma 1.3**) $q_{n+1} \geq q_n$, it follows that we have $\left|\theta - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n^2}$ for all convergents $\frac{p_n}{q_n}$. There are infinitely many such convergents[4], and hence we have infinitely many rational numbers satisfying inequality (10).

Conversely, suppose $\theta = \frac{a}{b}$ is rational. Suppose we have a rational number $\frac{p}{q} \neq \theta$ satisfying (10). Then, we have $\left|\frac{p}{q} - \theta\right| = \left|\frac{p}{q} - \frac{a}{b}\right| = \frac{|bp-aq|}{bq} \geq \frac{1}{bq}$ (since $\left|\frac{p}{q} - \theta\right| \neq 0$, so we must have $|bp - aq| > 0$).

Thus, we must have $q < b$, else we will have $\left|\frac{p}{q} - \theta\right| \geq \frac{1}{bq} \geq \frac{1}{q^2}$ which contradicts our assumption that $\frac{p}{q}$ satisfies inequality (20), and so there are only finitely many possible values of $q > 0$. Hence, there are only finitely many $\frac{p}{q}$ such that inequality (20) is satisfied (as there can only be a finite number of different p for fixed q). ■

And so we can now prove the following corollary to **theorem 1.5**:

COROLLARY **1.7**

θ is rational if and only if its continued fraction is finite, for any real number θ.

Equivalently, θ is irrational if and only if its continued fraction is infinite.

Proof.

We've already noted that, because of the closure of rational numbers under addition and multiplication, θ is rational if its continued fraction terminates. To show the converse, we use the result just proved in **lemma 1.6**, that for any rational θ, there are only finitely many rationals $\frac{p}{q}$ such that $\left|\theta - \frac{p}{q}\right| < \frac{1}{q^2}$. It follows that there are only finitely many convergents $\frac{p_n}{q_n}$ such that $\left|\theta - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n q_{n+1}}$. But $\frac{p_n}{q_n} \to \theta$. So it follows that we must have $\frac{p_n}{q_n} = \theta$ for some finite n. ■

---

[4] See **footnote 2**: if θ is irrational, its continued fraction is infinite, so there are infinitely many different convergents $\frac{p_n}{q_n} = [a_0, \ldots, a_n]$.

2. Rationals and irrationals, convergents and approximations

Having answered one of our initial questions – the issue of how finite and infinite continued fractions relate to rational and irrational numbers – let's see what kind of insights into them we can gain from this new way of looking at real numbers. We'll start with a result extrapolated from the recurrence relations in **lemma 1.3**, that will prove useful in this:

LEMMA 2.1

For any real number θ, we have

$$\theta = \frac{p_{n-1}\theta_n + p_{n-2}}{q_{n-1}\theta_n + q_{n-2}} \tag{11}$$

for all $n \geq 2$.

Proof.

Note first that

$$\theta = a_0 + \frac{1}{a_1 + 1/\theta_2} = \frac{a_0(a_1 + 1/\theta_2) + 1}{a_1 + 1/\theta_2} = \frac{a_0(a_1\theta_2 + 1) + \theta_2}{a_1\theta_2 + 1} = \frac{(a_0 a_1 + 1)\theta_2 + a_0}{a_1\theta_2 + 1} = \frac{p_1\theta_2 + p_0}{q_1\theta_2 + q_0} \tag{12}$$

(Recall, we define $p_0 = a_0$, $q_0 = 1$, $p_1 = a_0 a_1 + 1$ and $q_1 = a_1$.)

So, the proposition holds for $n = 2$. Suppose that it holds for $n = m$. Then, we get

$$\theta = \frac{p_{m-1}\theta_m + p_{m-2}}{q_{m-1}\theta_m + q_{m-2}} = \frac{p_{m-1}(a_m + 1/\theta_{m+1}) + p_{m-2}}{q_{m-1}(a_m + 1/\theta_{m+1}) + q_{m-2}} = \frac{(a_m p_{m-1} + p_{m-2})\theta_{m+1} + p_{m-1}}{(a_m q_{m-1} + q_{m-2})\theta_{m+1} + q_{m-1}} = \frac{p_m\theta_{m+1} + p_{m-1}}{q_m\theta_{m+1} + q_{m-1}} \tag{13}$$

(Note the substitution $\theta_m = a_m + \frac{1}{\theta_{m+1}}$, which follows from the Continued Fraction Algorithm.)

So the proposition holds for $n = m + 1$, and hence by the induction hypothesis for all $n \geq 2$. ■

With this result in hand, let's begin by looking at relationship between continued fractions and the rationals. We know that rationals have finite continued fractions, and thus that when we perform the Continued Fraction Algorithm on a rational the process will terminate – there exists N such that the convergent $\frac{p_N}{q_N}$ is equal to θ. Moreover, it can be observed that the convergents produced using the recursive equations proven in **lemma 1.3** are coprime[5]. So the Continued Fraction Algorithm provides a way of finding a rational in lowest terms – and thus, of finding the highest common factor of said rational. But of course there's another, much older algorithm that does the same – Euclid's Algorithm. Is there a relationship?

---

[5] Note that $\text{hcf}(p_0, q_0) = \text{hcf}(a_0, 1) = 1$, and $\text{hcf}(p_1, q_1) = \text{hcf}(a_0 a_1 + 1, a_1) = 1$. So it's true for the cases $n = 0$ and $n = 1$. Now assume it's true for $n = m - 1$, $m - 2$ and use the equations from **lemma 1.3** to show it's true for $n = m$. The result follows by induction.

Perhaps unsurprisingly, the answer is Yes. Here's how it works:

Let $\theta = \frac{h}{k} = [a_0, a_1, \ldots, a_N]$, where h, k are integers, $k \neq 0$, and N is finite. We know that $\frac{h}{k} = a_0 + \frac{1}{\theta_1}$, where θ_1 is rational, and so $h = a_0 k + \frac{1}{\theta_1} k$. Since h is an integer, we require $\theta_1 = \frac{k}{k_1}$, where k_1 is an integer.

Using the Continued Fraction Algorithm to find the value of θ_1, we find that $\frac{k}{k_1} = \theta_1 = a_1 + \frac{1}{\theta_2}$, with θ_2 rational, and so $k = a_1 k_1 + \frac{1}{\theta_2} k_1$, and therefore $\theta_2 = \frac{k_1}{k_2}$, where k_2 is an integer.

… and so on, until we get to the point where we have $\frac{k_{N-2}}{k_{N-1}} = \theta_{N-1} = a_{N-1} + \frac{1}{\theta_N}$, from which we get (in the same manner as above) $\frac{k_{N-1}}{k_N} = \theta_N$. Now, since the continued fraction process for $\frac{h}{k}$ terminates with a_N, we know that we must have $a_N = \theta_N$. So, $\frac{k_{N-1}}{k_N} = a_N$, and hence $k_{N-1} = k_N a_N$.

So what have we got all told? A whole series of expressions,

$$
\begin{aligned}
&h = a_0 k + \tfrac{1}{\theta_1} k = a_0 k + \tfrac{1}{k/k_1} k = a_0 k + k_1 \\
&k = a_1 k_1 + \tfrac{1}{\theta_2} k_1 = a_1 k_1 + k_2 \\
&k_1 = a_2 k_2 + k_3 \\
&\vdots \\
&k_{N-2} = a_{N-1} k_{N-1} + k_N \\
&k_{N-1} = a_N k_N
\end{aligned}
$$

… the exact same series of expressions we get from performing Euclid's Algorithm, and from which we conclude that $k_N = \text{hcf}(h, k)$. Note that we could go the other way too: using Euclid's Algorithm to find the partial quotients a_i. It's quite an eye-opener to realise that this ancient mathematical construct contained within it the seeds of a much more recent mathematical idea: the Continued Fraction Algorithm can be seen as a generalisation of Euclid's Algorithm that allows us to deal with those numbers that the ancient Greeks considered all-but-impossible to deal with, the irrational numbers.

Let's begin by defining a class of irrationals whose continued fractions have a particularly remarkable property.

DEFINITION 2.2

A *quadratic irrational* is an irrational number that is the root of a polynomial

$$ax^2 + bx + c$$

where a, b and c are integers such that $b^2 - 4ac$ is positive and is not a perfect square[6].

What is this remarkable property? Well, recall that in the previous section we asked: what if an (infinitely) continued fraction is recurrent (or periodic), that is, there exist integers k, t such that for all $n \geq k$, we have $a_{n+t} = a_n$?

We'll use the notation

$$\theta = [a_0, a_1, \ldots, a_{k-1}, \overline{a_k, a_{k+1}, \ldots, a_{k+t-1}}] \qquad (14)$$

to represent such recurrent continued fractions, where the sequence of partial quotients under the bar is repeated indefinitely, and with this in hand we'll proceed to prove the result the reader will have already guessed, that is,

THEOREM 2.3

An irrational number θ is quadratic if and only if its continued fraction is ultimately periodic, that is, of the form given in (14) above.

Proof.

First, given θ as in (13), define $\varphi = [\overline{a_k, \ldots, a_{k+t-1}}]$ and note that $\varphi = \theta_k$. So, by **lemma 2.1,** we have (for $k \geq 2$),

$$\theta = \frac{p_{k-1}\varphi + p_{k-2}}{q_{k-1}\varphi + q_{k-2}} \qquad (15)$$

(where the $\frac{p_i}{q_i}$ are convergents to θ). And, denoting by $\frac{p'_i}{q'_i}$ the convergents to φ, we also have (for $t \geq 2$)[7],

$$\varphi = \frac{p'_{t-1}\varphi + p'_{t-2}}{q'_{t-1}\varphi + q'_{t-2}} \qquad (16)$$

and hence we have,

$$q'_{t-1}\varphi^2 + (q'_{t-2} - p'_{t-1})\varphi - p'_{t-2} = 0 \qquad (17)$$

[6] We require these constraints because if $b^2 - 4ac$ were negative, the roots would be complex; if it were zero or a perfect square, the roots would be rational.

[7] If we have $t < 2$ ($\Rightarrow t = 1$ as we cannot have $t = 0$), *i.e.* $\varphi = [\overline{a_k}] = [a_k, a_k, a_k, \ldots]$, we can 'cheat' because obviously $[\overline{a}] = [\overline{a, a}] = [\overline{a, a, a}] = \cdots$ and so we can pick t to be any positive integer for the purposes of this proof.

So, φ is the root of a quadratic polynomial with integer coefficients, and hence is a quadratic irrational. From the equality given in (15), it follows that θ must therefore be quadratic.

For the case where $k < 2$, note that then we have $\theta = \text{integer} + \varphi$ and hence, as φ is quadratic, so is θ. So if θ has an ultimately periodic continued fraction, it is quadratic.

The proof of the converse is a little trickier and we'll skip over some of the more complicated number/symbol-crunching[8]. Suppose that θ is a quadratic irrational, and hence is the root of some polynomial $f(x) = ax^2 + bx + c$ as in **definition 2.2**. Fix $f(x)$.

Now, because θ is irrational we can write it in the form $\theta = [a_0, a_1, \ldots, a_n, \ldots]$, and we also know that, from **lemma 2.1**, we have – assuming (without loss of generality since we desire only to prove that θ is *ultimately* periodic) that $n \geq 2$ –

$$\theta = \frac{p_{n-1}\theta_n + p_{n-2}}{q_{n-1}\theta_n + q_{n-2}} \tag{18}$$

where, recall, $\theta_n = [a_n, a_{n+1}, \ldots]$.

Substituting this into $f(x)$, we get another polynomial[9]

$$f_n(x) = A_n x^2 + B_n x + C_n \tag{19}$$

The idea of the proof is to show that there are only a finite possible number of triples (A_n, B_n, C_n), and hence we can find $n_1 \neq n_2 \neq n_3$ such that we have

$$(A_{n_1}, B_{n_1}, C_{n_1}) = (A_{n_2}, B_{n_2}, C_{n_2}) = (A_{n_3}, B_{n_3}, C_{n_3})$$
$$= (A, B, C) \text{ say.}$$

Hence θ_{n_1}, θ_{n_2}, θ_{n_3} are all roots of the polynomial $Ax^2 + Bx + C$. Since there are three of them, at least two of them must be equal, say $\theta_{n_1} = \theta_{n_2}$. Hence, the periodic fraction for θ must be continuous.

---

[8] The sketch proof given here is based on the more complete one given in Hardy & Wright's *An Introduction to the Theory of Numbers*, pp.144-5, should the reader wish to see a detailed proof.

[9] The coefficients of polynomial (18) are defined as follows:

$$\begin{aligned} A_n &= ap_{n-1}^2 + bp_{n-1}q_{n-1} + cq_{n-1}^2 \\ B_n &= 2ap_{n-1}p_{n-2} + b(p_{n-1}q_{n-2} + p_{n-2}q_{n-1}) + 2cq_{n-1}q_{n-2} \\ C_n &= ap_{n-2}^2 + bp_{n-2}q_{n-2} + cq_{n-2}^2 \end{aligned}$$

Note that we have $A_n \neq 0$ as otherwise we would have $a(\frac{p_{n-1}}{q_{n-1}})^2 + b\frac{p_{n-1}}{q_{n-1}} + c = 0$, i.e. $f(\frac{p_{n-1}}{q_{n-1}}) = 0$, and f cannot have a rational root as its other one, θ, is irrational.

What we are *really* doing is showing that there are only finitely many different θ_n, and hence we have $\theta_k = \theta_{k+t}$ for some positive integers k, t, so once again θ must have a periodic continued fraction. ■

Further investigation reveals a number of other results about different quadratic irrationals. For example, we might want to know under what conditions a (quadratic) irrational θ has a *purely* periodic continued fraction, that is, $\theta = [\overline{a_0, a_1, \ldots, a_{t-1}}]$. It turns out that θ is purely periodic if and only if we have $\theta > 1$ and the conjugate θ' of θ (that is, the other root of the quadratic $ax^2 + bx + c$ that defines θ) satisfies $-1 < \theta' < 0$.

A corollary to this is that for any positive integer d that is not a perfect square, the continued fractions of $\sqrt{d} + [\sqrt{d}]$ and $\frac{1}{\sqrt{d} - [\sqrt{d}]}$ are both purely periodic, and so we can write $\sqrt{d}$ in the form [integer + purely periodic continued fraction], *i.e.*,

$$\sqrt{d} = [a_0, \overline{a_1, \ldots, a_t}] \tag{20}$$

for some integer $t \geq 1$. Not a purely periodic continued fraction itself, unfortunately, but very nearly. This result proves useful in trying to solve Diophantine Equations, in particular in finding the solutions to Pell's Equation.

We'll skip the proofs of these results, but they can be found in Alan Baker's *A Concise Introduction to the Theory of Numbers*, p.50, should the reader wish to view them. Instead, let's push on with some more general results relating to continued fractions and irrational numbers. In particular, we'll examine some issues relating to the convergents, and the issue of approximating real numbers.

Let's start by proving a couple of results that will prove useful here.

LEMMA 2.4

For any real number θ, we have

$$\left|\theta - \frac{p_n}{q_n}\right| = \frac{1}{q_n(\theta_{n+1} q_n + q_{n-1})} \tag{21}$$

Proof.

By **lemma 2.1** we have that $\theta = \frac{p_n \theta_{n+1} + p_{n-1}}{q_n \theta_{n+1} + q_{n-1}}$, and hence,

$$\left|\theta - \frac{p_n}{q_n}\right| = \left|\frac{p_n \theta_{n+1} + p_{n-1}}{q_n \theta_{n+1} + q_{n-1}} - \frac{p_n}{q_n}\right| = \left|\frac{p_{n-1} q_n - p_n q_{n-1}}{q_n (q_n \theta_{n+1} + q_{n-1})}\right| \tag{22}$$

and since by **lemma 1.4** we have $p_n q_{n+1} - p_{n+1} q_n = (-1)^{n+1}$, it follows that

$$\left|\theta - \frac{p_n}{q_n}\right| = \left|\frac{(-1)^{n+1}}{q_n (q_n \theta_n + q_{n-1})}\right| = \frac{1}{q_n (q_n \theta_n + q_{n-1})} \tag{23}$$

as required. ■

THEOREM 2.5

For any real number θ, given any two consecutive convergents $\frac{p_n}{q_n}$, $\frac{p_{n+1}}{q_{n+1}}$, at least one of them satisfies the inequality

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{2q^2} \tag{24}$$

(Note that this means that if θ is irrational, infinitely many convergents satisfy this inequality.)

Furthermore, if a rational $\frac{p}{q}$ satisfies this inequality, then it is a convergent to θ.

Proof.

Assume for a contradiction that we have a pair of consecutive convergents $\frac{p_n}{q_n}$, $\frac{p_{n+1}}{q_{n+1}}$ such that neither of them satisfies inequality (24), and so $\left|\theta - \frac{p_n}{q_n}\right| + \left|\theta - \frac{p_{n+1}}{q_{n+1}}\right| \geq \frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2}$.

Now, recall from the proof of **theorem 1.5** we noted that, for any two consecutive convergents $\frac{p_n}{q_n}$, $\frac{p_{n+1}}{q_{n+1}}$, θ lies between the two, and thus $\left|\theta - \frac{p_n}{q_n}\right|$, $\left|\theta - \frac{p_{n+1}}{q_{n+1}}\right|$ have opposite signs. Thus,

$$\left|\theta - \frac{p_n}{q_n}\right| + \left|\theta - \frac{p_{n+1}}{q_{n+1}}\right| = \left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right| = \frac{|p_n q_{n+1} - p_{n+1} q_n|}{q_n q_{n+1}} = \frac{1}{q_n q_{n+1}} \tag{25}$$

(using **lemma 1.4** for the last part of this calculation).

Time out for a moment to note a result we're going to use next: for any real numbers α, β with $\alpha \neq \beta$, we have

$$\alpha\beta < \tfrac{1}{2}(\alpha^2 + \beta^2) \tag{26}$$

It's easy to prove: note that $0 < (\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2$, and the result follows. Returning to our subject, if we take $\alpha = \frac{1}{q_n}$ and $\beta = \frac{1}{q_{n+1}}$, it follows from (26) that

$$\left|\theta - \frac{p_n}{q_n}\right| + \left|\theta - \frac{p_{n+1}}{q_{n+1}}\right| = \frac{1}{q_n q_{n+1}} < \frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2} \tag{27}$$

which contradicts our original assumption. Hence, at least one of $\frac{p_n}{q_n}$, $\frac{p_{n+1}}{q_{n+1}}$ must satisfy inequality (24).

To prove the converse, that if $\frac{p}{q}$ satisfies (24) then it is a convergent to θ, we will use *one* result without proof, namely that if $\theta = \frac{P\omega + R}{Q\omega + S}$, where $\omega > 1$ and P, Q, R, S are integers such that $Q > S > 0$ and $PS - QR = \pm 1$, then $\frac{R}{S}$ and $\frac{P}{Q}$ are two consecutive convergents to θ. This result is Theorem 172 of Hardy & Wright's book (pp.140-1), if the reader would like to see the proof.

Note that for $\frac{p}{q}$ satisfying (24), we have $\frac{p}{q} - \theta = \frac{\tau\xi}{q^2}$, where $\tau = \pm 1$ and $0 < \xi < \frac{1}{2}$, and that we can express $\frac{p}{q}$ as a continued fraction $[a_0, a_1, \ldots, a_n]$. We can also write $\theta = \frac{p_n\omega + p_{n-1}}{q_n\omega + q_{n-1}}$ where ω is some arbitrary real and $\frac{p_{n-1}}{q_{n-1}}$, $\frac{p_n}{q_n}$ are (respectively, the penultimate and ultimate) convergents to $\frac{p}{q}$.

Time out for another result: we can arbitrarily choose n to be odd or even. Why? Suppose $a_n \geq 2$. Then, note that $[a_0, a_1, \ldots, a_n] = [a_0, a_1, \ldots, a_n - 1, 1]$. Alternatively, if $a_n = 1$, then $[a_0, a_1, \ldots, a_n] = [a_0, a_1, \ldots, a_{n-1}, 1] = [a_0, a_1, \ldots, a_{n-2}, a_{n-1}+1]$. (In other words, we can arbitrarily add or subtract one from the number of quotients.) The upshot being that we can define, without loss of generality, $\tau = (-1)^{n-1}$.

Hence,

$$\frac{p_n}{q_n} - \theta = \frac{p_n q_{n-1} - p_{n-1} q_n}{q_n(q_n\omega + q_{n-1})} = \frac{(-1)\cdot(-1)^n}{q_n(q_n\omega + q_{n-1})} = \frac{(-1)^{n-1}}{q_n(q_n\omega + q_{n-1})}$$

i.e. $\frac{\tau\xi}{q_n^2} = \frac{(-1)^{n-1}\xi}{q_n^2} = \frac{(-1)^{n-1}}{q_n(q_n\omega + q_{n-1})}$, so $\frac{\xi}{q_n^2} = \frac{1}{q_n(q_n\omega + q_{n-1})}$ and from this we get $\omega = \frac{1}{\xi} - \frac{q_{n-1}}{q_n}$. Now, $0 < \xi < \frac{1}{2}$, so $\frac{1}{\xi} > 2$, and from the recurrence equations in **lemma 1.3** we have $q_n > q_{n-1}$ (meaning $\frac{q_{n-1}}{q_n} < 1$), and thus $\omega > 1$. Hence, from the assumed result mentioned earlier, taking $P = p_n$, $R = p_{n-1}$, $Q = q_n$ and $S = q_{n-1}$, we have that $\frac{p_{n-1}}{q_{n-1}}$, $\frac{p_n}{q_n}$ are two (consecutive) convergents to θ, and so $\frac{p}{q} = \frac{p_n}{q_n}$ is a convergent to θ. ■

What have we shown here? Recall that from **lemma 1.6** we know that for all convergents $\frac{p_n}{q_n}$ we have $\left|\theta - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n^2}$, for any real θ. If θ is irrational, we know that there are infinitely many convergents. What we've done in **theorem 2.5** is to show the stronger condition that, for irrational θ, there are still infinitely many convergents satisfying $\left|\theta - \frac{p_n}{q_n}\right| \leq \frac{1}{2q_n^2}$. This leads us to ask: given irrational θ, what is the largest constant κ we can find such that we have infinitely many convergents satisfying the inequality

$$\left|\theta - \frac{p_n}{q_n}\right| \leq \frac{1}{\kappa q_n^2} \qquad (28)$$

Another question we may wish to ask when considering the convergents is, *how quickly can we find a good rational approximation to some real number θ?* We know that, for any arbitrary real number ε, we can find infinitely many rational numbers $\frac{p}{q}$ such that $\left|\frac{p}{q} - \theta\right| < \varepsilon$. But how *fast* can we find such a rational? This is particularly important in the case of irrational θ because in real life we can never get a completely accurate calculation of an irrational, so being able to find a good approximation quickly is essential.

We've already shown, although not remarked upon it, that the convergents to θ are capable of giving us a good approximation to θ, since (as we've already noted) we know that all convergents $\frac{p_n}{q_n}$ satisfy $\left|\theta - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n^2}$. In other words, given arbitrary ε as above, all we have to do is find n such that $\frac{1}{q_n^2} < \varepsilon$, and we know that $\frac{p_n}{q_n}$ will give us the desired approximation. The question of 'How fast can we find such an approximation?' thus becomes a question of, 'How fast can we get such a q_n?'

Recalling the recurrence equation for q_n proven in **lemma 1.3**, that $q_n = a_n q_{n-1} + q_{n-2}$, it becomes apparent that the partial quotients a_n are the deciding factor in this, since q_n will increase faster the larger the values of a_n. Hardy and Wright, on p.163 of *An Introduction to the Theory of Numbers*, note that the following inequality holds[10]:

$$\left|\frac{p_n}{q_n} - \theta\right| < \frac{1}{a_{n+1}q_n^2} \qquad (29)$$

So, it follows that the larger the value of a_{n+1}, the better the approximation, and so we're in a position to answer *both* our questions: the best *general* approximation we can get to arbitrary real θ can be found by examining the special case

$$\theta = \frac{1}{1+}\frac{1}{1+}\cdots\frac{1}{1+}\cdots = \frac{1}{2}(\sqrt{5}-1) \qquad (30)$$

and in addition, the convergents to this value of θ will converge slower to an arbitrarily close approximation than those of any other value of θ.

How good is this approximation? First, note that the convergents to this θ are,

$$\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \ldots$$

and so on, so we get $q_{n-1} = p_n$ for all n. Now, using the result proven in **lemma 2.4**, and noting that $\theta_{n+1} = 1 + \frac{1}{1+}\frac{1}{1+}\cdots\frac{1}{1+}\cdots = 1+\theta$ for all n, we have that

$$\left|\frac{p_n}{q_n} - \theta\right| = \frac{1}{q_n(\theta_{n+1}q_n + q_{n-1})} = \frac{1}{q_n((1+\theta)q_n + q_{n-1})} = \frac{1}{q_n^2}\left(1+\theta+\frac{q_{n-1}}{q_n}\right)^{-1} = \frac{1}{q_n^2}\left(1+\theta+\frac{p_n}{q_n}\right)^{-1}$$

and hence, as $n \to \infty$, $\frac{p_n}{q_n} \to \theta$ and so we get

$$\left|\frac{p_n}{q_n} - \theta\right| \sim \frac{1}{q_n^2}\cdot\frac{1}{1+2\theta} = \frac{1}{q_n^2\sqrt{5}} \qquad (31)$$

This suggests that for all irrational θ, the largest constant κ such that infinitely many convergents to θ satisfy $\left|\theta - \frac{p_n}{q_n}\right| \leq \frac{1}{\kappa q_n^2}$ is $\kappa = \sqrt{5}$.

Let's prove it. We'll begin with,

[10] The proof of this follows from the result shown in **lemma 2.4**: it's easy to show that the right-hand-side of the equality is less than $\frac{1}{a_{n+1}q_n^2}$.

THEOREM 2.6 **(Hurwitz' Theorem)**

For any real number θ, given any three consecutive convergents, at least one of them satisfies the inequality

$$\left|\frac{p}{q}-\theta\right|<\frac{1}{q^2\sqrt{5}} \tag{32}$$

Consequently, if θ is irrational, infinitely many convergents satisfy this inequality.

Proof.

We've just done a whole load of number-crunching so we'll just sketch the outline of this proof. The complete proof can be found in Hardy & Wright's book (pp.164-5).

First, we define a neat shorthand, $\frac{q_{n-1}}{q_n}=b_{n+1}$, and then note that by **lemma 2.4** we have,

$$\left|\frac{p_n}{q_n}-\theta\right|=\frac{1}{q_n(\theta_{n+1}q_n+q_{n-1})}=\frac{1}{q_n^2}\cdot\frac{1}{\theta_{n+1}+{}^{q_{n-1}}\!/_{q_n}}=\frac{1}{q_n^2}\cdot\frac{1}{\theta_{n+1}+b_{n+1}}$$

So, to prove the theorem, we have to show that the inequality

$$\theta_i+b_i\le\sqrt{5} \tag{33}$$

cannot hold for all three of the consecutive values $i = n-1, n, n+1$. To do this, we suppose that it holds for $i = n-1, n$, and show that if it holds for $i = n+1$ then we get a contradiction, specifically that $a_n < 1$, which we know from the Continued Fraction Algorithm cannot be true for $n \ge 1$. The result follows. ■

We now show that this is the 'best possible' result in general – *i.e.* that $\kappa=\sqrt{5}$ is the largest possible value of κ such that we have infinitely many convergents to an irrational θ satisfying $\left|\theta-\frac{p_n}{q_n}\right|\le\frac{1}{\kappa q_n^2}$.

LEMMA 2.7

The largest possible value of possible value of κ such that *for all irrational* θ we have infinitely many convergents to θ satisfying $\left|\theta-\frac{p_n}{q_n}\right|\le\frac{1}{\kappa q_n^2}$ is $\kappa=\sqrt{5}$.

Proof.

Suppose for a contradiction that there exists $\kappa>\sqrt{5}$ such that for all irrational θ we have infinitely many convergents satisfying $\left|\theta-\frac{p_n}{q_n}\right|\le\frac{1}{\kappa q_n^2}$.

Fix $\theta = \frac{1}{2}(\sqrt{5}-1) = \frac{1}{1+}\frac{1}{1+}\cdots\frac{1}{1+}\cdots$. By our assumption, we can find infinitely many convergents satisfying $\left|\theta - \frac{p_n}{q_n}\right| \le \frac{1}{\kappa q_n^2}$, *i.e.* infinitely many $\frac{p}{q}$ satisfying,

$$\theta = \tfrac{1}{2}(\sqrt{5}-1) = \tfrac{p}{q} + \tfrac{\delta}{q^2}$$

where $|\delta| < \frac{1}{\kappa} < \frac{1}{\sqrt{5}}$.

From this we get $\frac{\delta}{q} = q \cdot \frac{1}{2}(\sqrt{5}-1) - p$, and thus $\frac{\delta}{q} - \frac{1}{2}q\sqrt{5} = -\frac{1}{2}q - p$. Squaring both sides gives us $\frac{\delta^2}{q^2} - \delta\sqrt{5} + \frac{5}{4}q^2 = \frac{1}{4}(p+q)^2$, and hence,

$$\tfrac{\delta^2}{q^2} - \delta\sqrt{5} = p^2 + pq - q^2$$

Note that since $|\delta| < \frac{1}{\sqrt{5}}$, the left hand side of this equation is less than 1. It follows that for there to be infinitely many convergents $\frac{p}{q}$ satisfying $\left|\theta - \frac{p_n}{q_n}\right| \le \frac{1}{\kappa q_n^2}$, we require infinitely many $\frac{p}{q}$ such that $p^2 + pq - q^2 = 0$, with p, q integers, which is impossible. So there are only finitely many convergents to $\theta = \frac{1}{2}(\sqrt{5}-1)$ satisfying $\left|\theta - \frac{p_n}{q_n}\right| \le \frac{1}{\kappa q_n^2}$, and hence our original assumption is false, so our proof is complete. ■

This is in a sense a disappointing result, because it places a limit on how quickly we can find an approximation to a general irrational θ, and we might have hoped to do better.

The thought might have occurred to the reader: *but what if we were to exclude* $\frac{1}{2}(\sqrt{5}-1)$ *from the numbers we consider?* After all, it's only one number, as opposed to the infinitely many other numbers we may want to approximate. Can we get a better general approximation if we make this exclusion?

The answer is Yes, although we have to exclude not just $\frac{1}{2}(\sqrt{5}-1)$ but also any irrationals equivalent to it (that is, any irrationals whose continued fractions have the form $[a_0, \ldots, a_{k-1}, 1, 1, 1, \ldots]$). In fact in this case we can in general find infinitely many convergents to θ satisfying the inequality

$$\left|\theta - \tfrac{p_n}{q_n}\right| \le \tfrac{1}{q_n^2\sqrt{8}} \tag{34}$$

The best result occurs when we consider only θ where the partial quotients a_i are unbounded, and we'll finish off this section with the proof of this.

LEMMA 2.8

Given any irrational θ with unbounded partial quotients a_i, and any real $\varepsilon > 0$, there are infinitely many convergents to θ satisfying the inequality

$$\left|\theta - \frac{p}{q}\right| < \frac{\varepsilon}{q^2} \tag{35}$$

Proof.

Given an irrational θ with unbounded partial quotients, recall from **lemma 2.4** that we have $\left|\frac{p_n}{q_n} - \theta\right| = \frac{1}{q_n(\theta_{n+1}q_n + q_{n-1})}$ for all convergents $\frac{p_n}{q_n}$, and hence we have $\left|\frac{p_n}{q_n} - \theta\right| < \frac{1}{a_{n+1}q_n^2}$ (see **footnote 10**). Since the a_i are unbounded we can find infinitely many n such that $\frac{1}{a_{n+1}} < \varepsilon$, and so $\left|\frac{p_n}{q_n} - \theta\right| < \frac{1}{a_{n+1}q_n^2} < \frac{\varepsilon}{q_n^2}$. ■

3. Titbits

We'll conclude this essay with a look at a few interesting examples and applications of continued fractions. These aren't necessarily trivial examples (Pell's Equation certainly isn't) but we'll consider them in a simple and non-rigorous style, hence the title!

EXAMPLE 3.1 **(Pell's Equation)**

Pell's Equation is a simple but tricky example of a Diophantine Equation,

$$x^2 - dy^2 = 1 \tag{36}$$

where d is a positive integer that is not a perfect square.

Simple for obvious reasons, but why tricky? Because finding non-trivial solutions, *i.e.* a solution other than $x = \pm 1$, $y = 0$, is surprisingly difficult. Continued fractions prove highly useful in finding solutions.

Suppose that x, y are positive integers satisfying $x^2 - dy^2 = 1$. Dividing through by $x + y\sqrt{d}$ we get $x - y\sqrt{d} = \frac{1}{x+y\sqrt{d}} > 0$, so $x > y\sqrt{d}$ and thus $x - y\sqrt{d} < \frac{1}{y\sqrt{d}+y\sqrt{d}}$. Hence,

$$\left|\frac{x}{y} - \sqrt{d}\right| < \frac{1}{2y^2\sqrt{d}} < \frac{1}{2y^2} \tag{37}$$

So, by **theorem 2.5**, $\frac{x}{y}$ is a convergent to $\sqrt{d}$. So, write $\theta = d$, and consider all our notation to be as usual in the previous chapters, with $\frac{p_i}{q_i}$ the convergents as usual, so that $\frac{x}{y} = \frac{p_n}{q_n}$ for some fixed n, and so $p_n^2 - dq_n^2 = 1$ and also, by **lemma 2.1,**

$$\sqrt{d} = \theta = \frac{p_n\theta_{n+1} + p_{n-1}}{q_n\theta_{n+1} + q_{n-1}}$$

A little calculation reveals that we require n to be odd, else we would have $p_n < q_n\sqrt{d}$, *i.e.* $x < y\sqrt{d}$, which contradicts what we know about x, y. In fact, we require $n = lm - 1$, where $l = 1, 2, 3, \ldots$ when m is even, and $l = 2, 4, 6, \ldots$ when m is odd, and it turns out that, for x, y positive integers, x, y are solutions to the Pell equation if and only if $x = p_n$, $y = q_n$, where $\frac{p_n}{q_n}$ is a convergent to $\sqrt{d}$ with $n = lm - 1$ as before.

EXAMPLE 3.2 **(A few of Euler's Continued Fractions)**

In the first book of Euler's *Introduction to Analysis of the Infinite*, the final chapter is dedicated to continued fractions. Euler describes them slightly differently from how they were defined in this essay. In particular, he allows for a more general type of continued fraction, of the form,

$$\theta = a + \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \ddots}}} \tag{38}$$

There are positive and negative aspects to this. The nice thing about the continued fraction algorithm and continued fractions as defined in this essay is that, as Alan Baker notes, *It sets up a one-to-one correspondence between all irrational* θ *and all infinite sets of integers* $[a_0, a_1, \ldots]$ *with the* a_i *positive for* $i \geq 1$*, and also between all rational* θ *and all finite sets of integers* $[a_0, a_1, \ldots, a_n]$*, again with the* a_i *positive for* $1 \leq i \leq n$*.* One-to-one correspondence is a very useful mathematical tool, especially if it provides us with a link to the (uncountably many) irrationals.

On the other hand, allowing continued fractions of this form produces some fascinating patterns. For example, Euler gives

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \ddots}}}}} \tag{39}$$

Also,

$$\log 2 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \ddots}}}}} \tag{40}$$

And,

$$\frac{1}{e-1} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\ddots}}}}}} \tag{40}$$

These patterns are so simple, for such seemingly complicated numbers, that it's immediately obvious that they could be extremely useful. When we're approximating general irrationals, as in this essay, it's simpler to use continued fractions where the numerators are all 1's, but for specific numbers like these, it can be useful to consider the more general type of continued fraction.

EXAMPLES 3.3 **(Real titbits!)**

Other applications of continued fractions, in and outside of number theory, include:

- Other Diophantine equations besides Pell's equation, for example finding all solutions to *linear* Diophantine equations (*e.g.* $ax + by = c$).
- Solving congruences (*e.g.* $ax \equiv b \bmod c$)
- Applications to Chaos Theory. There are some notes on this by Robert M. Corless of the University of West Ontario Applied Maths Department at http://www.cecm.sfu.ca/publications/organic/confrac/confrac.html.

Bibliography

Alan Baker, *A Concise Introduction to the Theory of Numbers.*
(Cambridge University Press, 1984.)

Leonard Euler, *Introduction to Analysis of the Infinite.*
(Translated by John D. Blanton. Springer-Verlag, 1988.)

G.H. Hardy & E.M. Wright, *An Introduction to the Theory of Numbers.*
(5th Edition. Oxford University Press, 1979.)

http://www.cecm.sfu.ca/publications/organic/confrac/confrac.html

http://www.calvin.edu/academic/math/confrac/intro.html